Arrow Polynomial and Checkerboard Colorability

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Based on the paper: *On arrow polynomials of checkerboard colorable virtual links* by Qingying Deng , Xian'an Jin , Louis H. Kauffman

A remarks on Braids

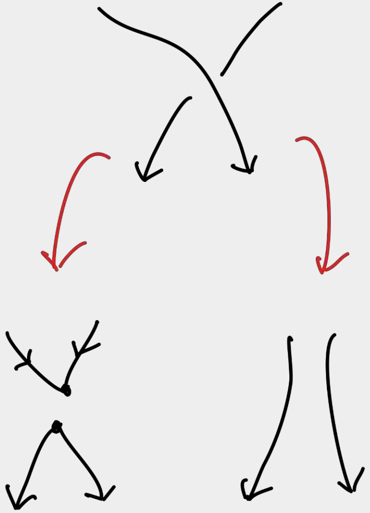
1 Why the braiding procedure is okay for checkerboard colorings?

2 They are not needed for the proof

3 They make the pictures “look nicer”

Recalling Definitions

Arrow polynomial smoothings

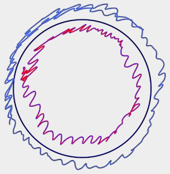


Disoriented Smoothing

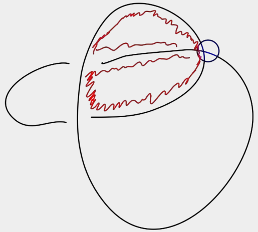
Oriented Smoothing

Diagram Colorability

Colorable



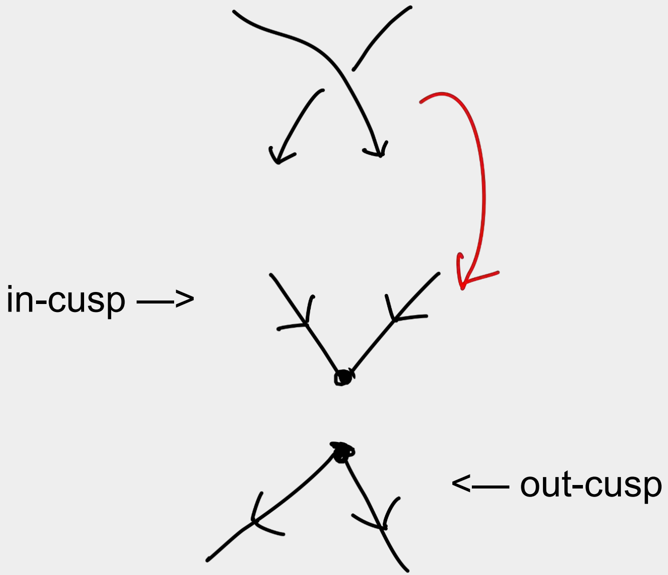
Not Colorable



Nodal Cusps

Definition

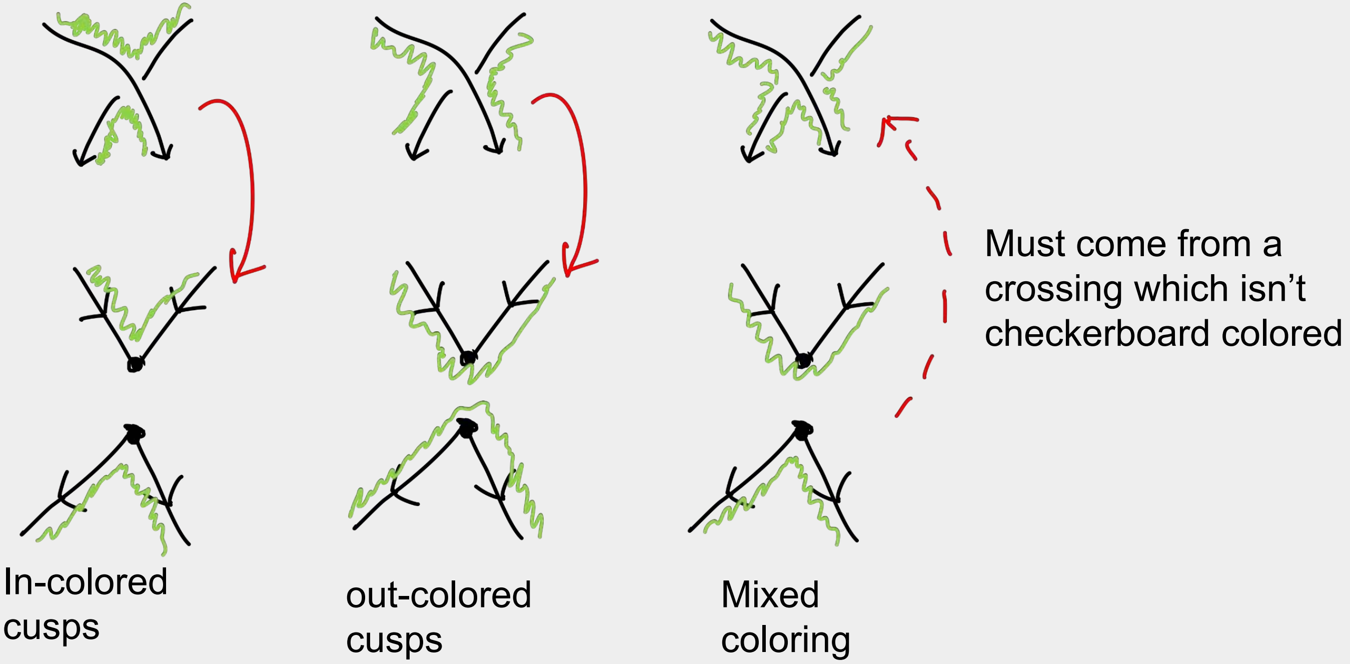
An *in-cusp* is a cusp with the arrows pointing in, and an *out-cusp* is a cusp with the arrows pointing out.



Coloring Nodal Cusps

Definition

An *in-coloring* is when the inside of the acute angle formed by a cusp is colored, similarly, an *out-coloring* is when the outside of the acute angle formed by a cusp is colored.



Circle Graphs

A state σ of some diagram D is given to us after we replace each classical crossing by its oriented or disoriented smoothing. The state σ will consist of many "circles" which have virtual crossings and cusps. These "circles" are what we call circle graphs.

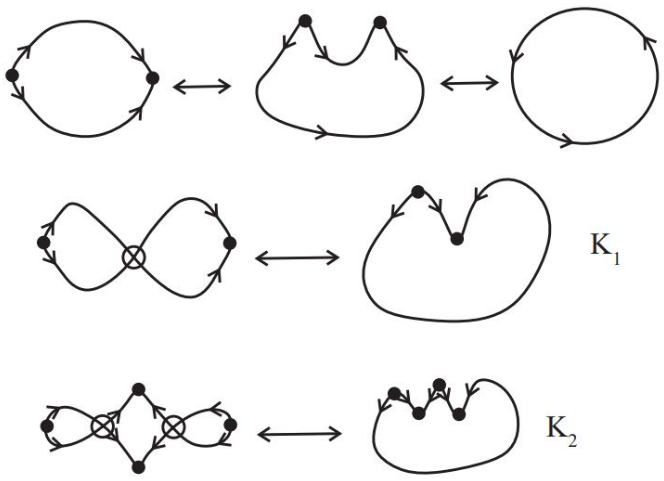


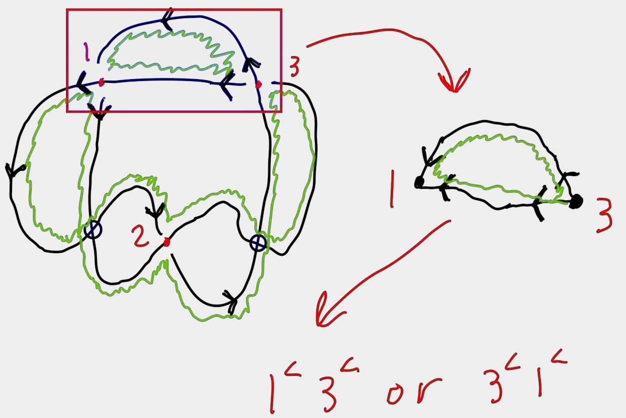
Fig. 9: Reduction rule for the arrow polynomial.

Graphic from Qingying Deng , Xian'an Jin , Louis H. Kauffman(2020)

“Word” Notation for Circle Graphs

We can express each circle graph of a state σ as a word with letters being of the form $i^<$ or $i^>$ with $i \in \mathbb{N}$. Each $i^<$ or $i^>$ represents a cusp corresponding to the classical crossing labeled i, with the superscript $<$ indicating that the cusp is in-colored and the super-script $>$ representing that the cusp is out-colored. In order to write a word start at any cusp and move clockwise along the circle graph.

Example



A state σ corresponds to a collection of words.

“Word” Notation for Circle Graphs

1. Words are cyclic, for example we have that $a^{<}b^{>}c^{<}d^{>} = d^{>}a^{<}b^{>}c^{<} = c^{<}d^{>}a^{<}b^{>} = b^{>}c^{<}d^{>}a^{<}$. This corresponds to starting at different cusps in a circle graph containing 4 cusps.

2. Assuming that the diagram is colored, each crossing must form cusps that are both in-colored or out-colored. So, a state σ cannot have two words, one which contains $i^{<}$ and another which contains $i^{>}$. (refer back to slide 5)

Examples: The state σ cannot contain both the words $a^{<}b^{>}c^{<}d^{>}$ and $b^{<}g^{>}$. The state σ cannot contain the word $b^{>}b^{<}$.

Claim 1

Claim

In any circle graph, every in-cusp is followed by an out-cusp and every out-cusp is followed by an in-cusp.

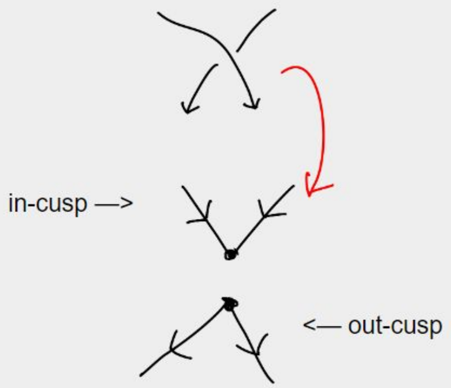
Why? An out-cusp was the exit of some crossing before applying disoriented smoothing, so, following either strand along the arrow must lead to the entrance of some crossing which became in-cusp. Similar logic applies to the in-cusps. (look at slide 6 again)

Claim 2

Claim

If $i^{<}Xi^{<}Y$ or $i^{>}Xi^{>}Y$ is a word of a circle graph of σ, then X and Y both have even length.

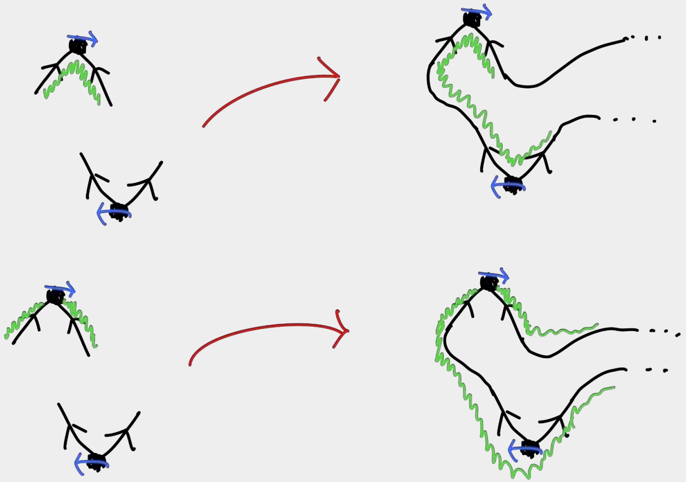
proof: By the diagram below, every crossing makes one in-cusp and one out-cusp when disoriented smoothing is applied. Hence, one of the cusps labeled i is an in-cusp and the other is an out-cusp. By Claim 1 X and Y must have even length in order to maintain alternating in-cusps and out-cusps.



Word Reducing Lemma

Lemma For a colored diagram, the word corresponding to some circle graph is reducible iff it contains the subword $a^{<}b^{<}$ or $a^{>}b^{>}$. If the word is $Xa^{<}b^{<}Y$ or $Xa^{>}b^{>}Y$ it can be reduced to XY.

proof: The word is reducible iff the circle graph is, meaning that we can find some adjacent pair of cusps that cancel. By the diagram below if two cusps are made to cancel by making nodal arrows face the same direction then both must be in-colored or both must be out-colored. A similar diagram works for the reverse direction.



Claim 3

Claim For any reduced word of σ, it is trivial or the labels are all different.

proof: Suppose that W is a nontrivial word containing two cusps labeled i. Without loss of generality assume that both cusps labeled i are in-colored. Then, we have $W = i^{<}Xi^{<}Y$. By Claim 2, X has even length and by the Word Reducing Lemma $X = 1^{<}2^{>}3^{<}\cdots 2k^{>}$ or $X = 1^{>}2^{<}3^{>}\cdots 2k^{<}$. So, $W = (i^{<}1^{<})2^{>}3^{<}\cdots 2k^{>}i^{<}Y$ or $W = i^{<}1^{>}2^{<}3^{>}\cdots(2k^{<}i^{<})Y$ both of which can be reduced by the Word Reducing Lemma.

K-degree example

Note that any summand of $\langle D \rangle_{NA}$ has the following form:

$$A^s(K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}).$$

Then the *k-degree* of this summand is defined to be

$$i_1 \times j_1 + i_2 \times j_2 + \cdots + i_v \times j_v,$$

The k-degree of a state σ is defined to be the k-degree of the summand corresponding to state σ.

Example If σ has the following circle graphs with multiplicity: d, K_1, K_2, K_2, K_5. Then, the k-degree of σ is $\deg_K(\sigma) = 1 + 2(2) + 5 = 10$.

Claim 4

Claim For the state σ, $\deg_K(\sigma) \equiv 0 \bmod 2$, meaning that its k-degree is even.

start of proof: Let σ_0 be the state obtained when oriented smoothing is chosen for each classical crossing. Then, no cusps are formed, so, $\deg_K(\sigma_0) = 0$. Let $\sigma = \sigma_n, \sigma_{n-1}, ..., \sigma_1, \sigma_0$, be a sequence of states where $\sigma_{\ell+1}$ differs from σ_ℓ by changing one disoriented smoothing to an oriented smoothing. We wish to show that $\deg_K(\sigma_{\ell+1}) \equiv \deg_K(\sigma_\ell) \bmod 2$. Let i be the classical crossing changed to a disoriented smoothing, and without loss of generality assume that both of its cusps are in-colored in state $\sigma_{\ell+1}$.

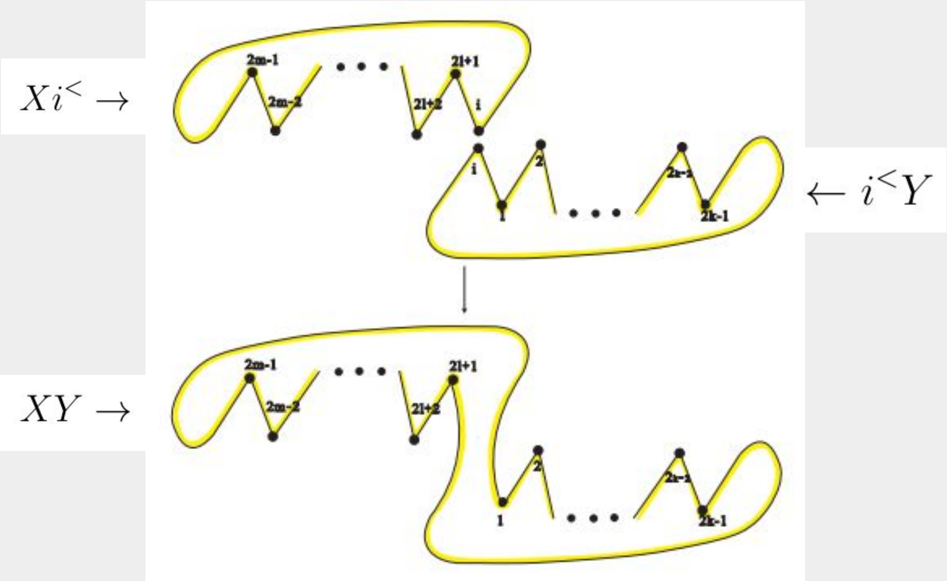
There are two possible cases:

Case 1 Both cusps $i^<$ belong to different circle graphs of $\sigma_{\ell+1}$

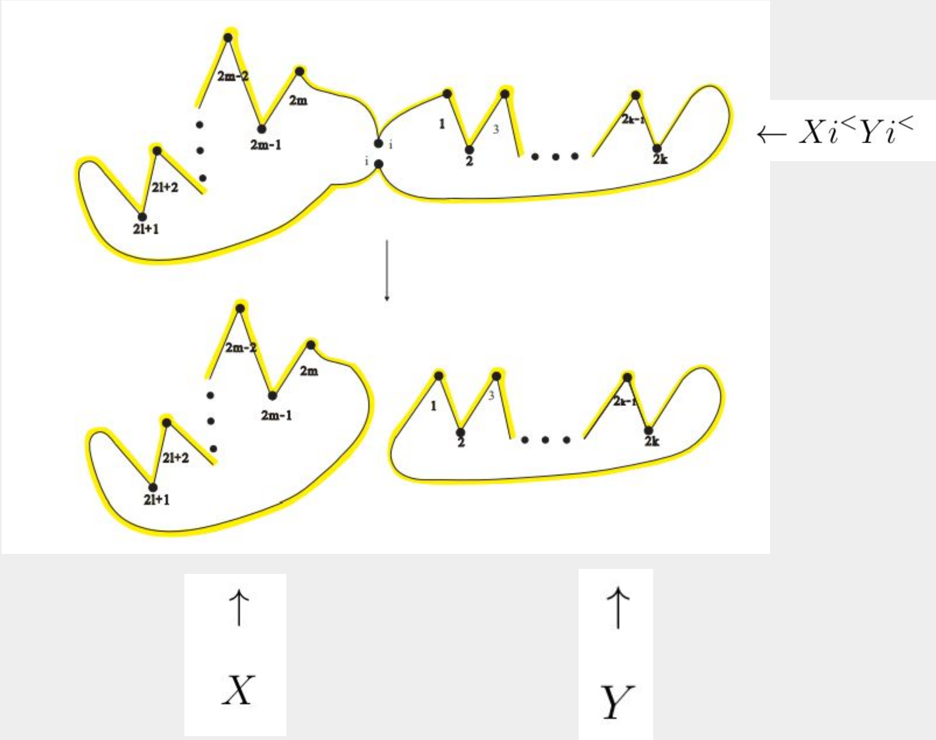
Case 2 Both cusps $i^<$ belong to the same circle graph of $\sigma_{\ell+1}$

Changing a smoothing examples

Case 1



Case 2



Diagrams from Qingying Deng , Xian'an Jin , Louis H. Kauffman (2020)

Claim 4 Case 1

For Case 1 each cusp $i^<$ of is a part of different circle graphs of $\sigma_{\ell+1}$ meaning that they are contained in separate words. The possible words are:

Case 1.1 $W_1 = i^<(a_1^> a_2^< \cdots a_{2k-1}^>)$ and $W_2 = i^<(b_1^> b_2^< \cdots b_{2m-1}^>)$

Case 1.2.1 $W_1 = i^<(a_1^< a_2^> \cdots a_{2k-1}^<)$ and $W_2 = i^<(b_1^> b_2^< \cdots b_{2m-1}^>)$

Case 1.2.2 $W_1 = i^<(a_1^> a_2^< \cdots a_{2k-1}^>)$ and $W_2 = i^<(b_1^< b_2^> \cdots b_{2m-1}^<)$

Case 1.3 $W_1 = i^<(a_1^< a_2^> \cdots a_{2k-1}^<)$ and $W_2 = i^<(b_1^< b_2^> \cdots b_{2m-1}^<)$

Notice that cases 1.2.1 and 1.2.2 are the same.

Claim 4 Case 1.1

Case 1.1 $W_1 = i^<(a_1^> a_2^< \cdots a_{2k-1}^>)$ and $W_2 = i^<(b_1^> b_2^< \cdots b_{2m-1}^>)$

By the Word Reducing Lemma, the words W_1 and W_2 are reduced, which means that the correspond to circle graphs K_k and K_m in $\sigma_{\ell+1}$. On the other hand, in σ_ℓ these circle graphs combine, the corresponding word is $(a_1^> a_2^< \cdots a_{2k-1}^>)(b_1^> b_2^< \cdots b_{2m-1}^>)$. By the Word Reducing Lemma, we reduce the word from the middle out as follows:

$$(a_1^> a_2^< \cdots a_{2k-1}^>)(b_1^> b_2^< \cdots b_{2m-1}^>)$$

$$= (a_1^> a_2^< \cdots a_{2k-2}^<)(a_{2k-1}^> b_1^>)(b_2^< \cdots b_{2m-1}^>)$$
$$= (a_1^> a_2^< \cdots a_{2k-2}^<)(b_2^< \cdots b_{2m-1}^>)$$

Repeatedly applying this reduction will cancel all a's or all b's. What remains is a reduced word of length $2m - 2k$ if all the a's cancel or a word of length $2k - 2m$ if all of the b's cancel, which corresponds to circle graph K_{m-k} or K_{k-m}. Hence we have that either:

$$\deg_K(\sigma_\ell) = \deg_K(\sigma_{\ell+1}) - k - m + (m - k) \equiv \deg_K(\sigma_{\ell+1}) \bmod 2$$

or

$$\deg_K(\sigma_\ell) = \deg_K(\sigma_{\ell+1}) - k - m + (k - m) \equiv \deg_K(\sigma_{\ell+1}) \bmod 2$$

Claim 4 Case 2

For Case 2 each cusp $i^<$ is part of the same circle graph of $\sigma_{\ell+1}$ meaning that they are contain in the same word. By claim 2, the subwords consisting of a's and b's must have even length. The possible words are:

Case 2.1.1 $W = i^<(a_1^> a_2^< \cdots a_{2k}^<) i^<(b_1^< b_2^> \cdots b_{2m}^>)$

Case 2.1.2 $W = i^<(a_1^< a_2^> \cdots a_{2k}^>) i^<(b_1^> b_2^< \cdots b_{2m}^<)$

Case 2.2.1 $W = i^<(a_1^< a_2^> \cdots a_{2k}^>) i^<(b_1^< b_2^> \cdots b_{2m}^>)$

Case 2.2.2 $W = i^<(a_1^> a_2^< \cdots a_{2k}^<) i^<(b_1^> b_2^< \cdots b_{2m}^<)$

Case 2.1.1 and Case 2.1.2 are analogous by cyclic permutation on the word W. Similarly, Case 2.2.1 is analogous to Case 2.2.2

Claim 4 Case 2.1.1

Case 2.1.1 $W = i^<(a_1^> a_2^< \cdots a_{2k}^<) i^<(b_1^< b_2^> \cdots b_{2m}^>)$

By the Word Reducing Lemma, the reduced word is $W = i^<(a_1^> a_2^< \cdots a_{2k}^<)(b_2^> \cdots b_{2m}^>)$, this word has length $2m + 2k$ which means that it corresponds to the circle graph K_{m+k} in $\sigma_{\ell+1}$. On the other hand, in σ_l the circle graph splits into two circle graphs, the corresponding words are $W_1 = (a_1^> a_2^< \cdots a_{2k}^<)$ and $W_2 = (b_1^< b_2^> \cdots b_{2m}^>)$. By the Word Reducing Lemma W_1, W_2 are reduced and so correspond to the circle graphs K_k and K_m. Hence, we have that:

$$\deg_K(\sigma_{\ell+1}) = \deg_K(\sigma_\ell) - (m + k) + k + m \equiv \deg_K(\sigma_\ell) \bmod 2$$

Theorem 4.3 (1)

We have shown Claim 4 which stated that a checkerboard colored diagram D only has states σ with even k-degree. It follows that the set of all k-degrees of summands of the arrow polynomial $\langle D \rangle_{NA}$ must be even.

Possible Ways Forward

1. New Invariant
2. Trying something with cables(every 2-cabled diagram is checkerboard colorable?)